TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 10

Exercise 10.1. Let G be a finite group.

(1) Define the concepts of a series and of a normal series in G.
(2) Exhibit an example of a series which is not normal.

Exercise 10.2.

(1) Define the concept of a right operator group (G, Ω, α).
(2) If G is an Ω-group (that is, there is a right operator group (G, Ω, α), for some α), define the concept of an Ω-subgroup of G and of an Ω-series in G.
(3) Define the concept of characteristic and fully invariant subgroups of a group.
(4) Exhibit examples of a group G and a subgroup $H \le G$ such that
    (a) H is not normal in G;
    (b) H is normal, but not characteristic in G;
    (c) H is characteristic, but not fully invariant in G.

Exercise 10.3. Let G be a group, $M \le N \le G$.

(1) Exhibit an example in which $M \trianglelefteq N$ and $N \trianglelefteq G$ but $M \ntrianglelefteq G$.
(2) Show that if M is characteristic in N and $N \trianglelefteq G$, then $M \trianglelefteq G$.
(3) Show that if M is characteristic in N and N is characteristic in G, then M is characteristic in G.

Exercise 10.4.

(1) Define the concept of a refinement of an Ω-series, and of an Ω-composition series of a finite group.
(2) Show that the quotients in a composition series of a group G (thus $\Omega = \emptyset$ here) are simple groups.
(3) Show that the quotients H_{i+1}/H_i of a principal series (thus $\Omega = \mathrm{Inn}(G)$ here)
$$1 = H_0 < H_1 < \cdots < H_n = G$$
are characteristically simple groups.
(4) Show that a minimal normal subgroup of a finite group G is a characteristically simple group.
(5) Show that a finite, characteristically simple group is the direct product of isomorphic copies of a simple group.

Exercise 10.5. Let G be a group. Consider the map
$$\iota : G \to G^G$$
$$g \mapsto (x \mapsto g^{-1}xg),$$
where G^G is the monoid of maps on G.

(1) Show that $\iota(g) \in \mathrm{Aut}(G)$ for each $g \in G$.

(2) Show that $\iota : G \to \mathrm{Aut}(G)$ is a morphism of groups, so that

$$\mathrm{Inn}(G) = \iota(G)$$

is a subgroup of $\mathrm{Aut}(G)$.

(3) Show that for $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ we have

$$\iota(g)^\alpha = \alpha^{-1}\iota(g)\alpha = \iota(g^\alpha),$$

so that $\mathrm{Inn}(G)$ is a *normal* subgroup of $\mathrm{Aut}(G)$.

(4) Show that if $N \trianglelefteq G$ and $\alpha \in \mathrm{Aut}(G)$, then $N^\alpha \trianglelefteq G$.