TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 9

Exercise 9.1. Let ρ be a reprsentation of the finite group G, and χ its character.
Prove that
$$\ker(\rho) = \{ g \in G : \chi(g) = \chi(1) \} .$$

Exercise 9.2. Let G be a finite group, and χ an irreducible character of G.
(1) Show that if $\chi(1) \neq 1$, then there is $g \in G$ such that $\chi(g) = 0$.
(2) Give an example to show that there is a finite group G, and a character χ of G such that $\chi(1) \neq 1$ and $\chi(g) \neq 0$ for all $g \in G$.
(HINT: Clearly such a χ cannot be irreducible.)

Exercise 9.3. Prove that the character table of a finite group is a non-singular matrix.

Exercise 9.4. Prove the "other" ortogonality relations, that is, if G is a finite group, and $x, y \in G$, then
$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(x)\chi(y^{-1}) = \begin{cases} |\, C_G(x) \,| & \text{if } x \text{ and } y \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 9.5. Give two proofs
(1) one using characters
(2) the other without using characters

that if G is a finite group, $N \trianglelefteq G$ and $x \in G$, then
$$|\, C_G(x) \,| \geq \left| C_{G/N}(xN) \right| .$$