TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 8

Exercise 8.1.

(1) Let $f : G \to \mathbf{C}$ be a class function, so that $\alpha = \sum_{g \in G} f(g)g \in Z(\mathbf{C}[G])$.

Let $\rho : G \to \mathrm{GL}(V_\chi)$ be an irreducible representation affording the irreducible character χ. Let

$$\begin{aligned} \varphi : \mathbf{C}[G] &\to \mathrm{End}(V_\chi) \\ \sum_{g \in G} f(g)g &\mapsto \sum_{g \in G} a(g)\rho(g) \end{aligned}$$

be the natural epimorphism of algebras.

Show that

$$\varphi\left(\sum_{g \in G} f(g)g\right)$$

is a scalar matrix which has on the diagonal the number

$$\frac{|\,G\,|}{\chi(1)} \cdot (f, \overline{\chi}).$$

(2) Show that if $\chi \in \mathrm{Irr}(G)$, and $f : G \to \mathbf{C}$ is a class function, whose values are algebraic integers, then

$$\frac{1}{\chi(1)} \sum_{g \in G} f(g)\chi(g)$$

is an algebraic integer.

(3) Show that if $\chi \in \mathrm{Irr}(G)$, then $\chi(1)$ divides the order of G.

Exercise 8.2.

(1) Let E be a subfield of $\mathbf{C}$, such that $E/\mathbf{Q}$ is a Galois extension. Suppose that $\mathrm{Gal}(E/\mathbf{Q})$ is abelian.

Show that for $\alpha \in E$ and $g \in \mathrm{Gal}(E/\mathbf{Q})$ we have $|\,\alpha^g\,| = |\,\alpha\,|^g$.

(2) (Optional, see the notes) Show that this need not hold when $\mathrm{Gal}(E/\mathbf{Q})$ is non-abelian.

Exercise 8.3.

(1) Let $z_1, \dots, z_n \in \mathbf{C}$ have all absolute value 1.

Show that is $|\,z_1 + \dots + z_n\,| = n$, then $z_1 = \dots = z_n$.

(2) Let $\omega_1, \dots, \omega_n \in \mathbf{C}$ be roots of unity.

Show that if

$$\alpha = \frac{\omega_1 + \dots + \omega_n}{n}$$

is an algebraic integer, then either $\alpha = 0$, or $\omega_1 = \dots = \omega_n = \alpha$.