TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 7

Exercise 7.1.

(1) Prove that the product of two characters is a character.
(2) Compute all products of the irreducible characters of S_3

Exercise 7.2. If χ is a character of the finite group G, define a function $\overline{\chi} : G \to \mathbf{C}$ by

$$\overline{\chi}(g) = \overline{\chi(g)}.$$

(1) Prove that $\overline{\chi}$ is a character of G.
(2) Prove that

$$\overline{\chi}(g) = \chi(g^{-1}).$$

Exercise 7.3. Let G be finite group acting on the finite set Ω.

(1) Define the associated permutation representation ρ and its character χ.
(2) Show that $\chi(g) = \text{Fix}(g) = |\{ \alpha \in \Omega : \alpha^g = \alpha \}|$ is the number of fixed points of g.
(3) Show that the numer of orbits of G on Ω is given by

$$\frac{1}{|G|} \sum_{g \in G} \text{Fix}(g) = (1, \chi),$$

where 1 denotes the trivial character.
(4) Show that G acts transitively on Ω (i.e., there is only one orbit) iff $\chi = 1 + \psi$, where ψ is a character such that $(1, \psi) = 0$.
(5) Define what is meant for G to act double transitively on Ω (one also says G acts 2-transitively, or that G is 2-transitive).
(6) Show that G is 2-transitive iff ψ is irreducible.

Exercise 7.4. Compute the character tables of S_3, A_4, S_4.

Exercise 7.5. Let R be a commutative, unital ring of characteristic zero, so that $\mathbf{Z}$ is a subring with unity of R.

(1) Show that for $\alpha \in R$, the following are equivalent:
    (a) there exists $n \geq 1$ and $a_1, \dots, a_n \in \mathbf{Z}$ such that

    $$\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0.$$

    (b) The subring

    $$\mathbf{Z}[\alpha] = \left\{ a_0 + a_1\alpha + \cdots + a_k\alpha^k : k \in \mathbf{N}, a_i \in \mathbf{Z} \right\}$$

    of R is finitely generated as a $\mathbf{Z}$-module.
    (c) The subring $\mathbf{Z}[\alpha]$ of R is contained in a subring of R whose additive group is a finitely generated $\mathbf{Z}$-submodule of R.

An element α satisfying these conditions is said to be *integral*. If $R = \mathbf{C}$, then α is said to be an *algebraic integer*.

(2) Show that the integral elements of R form a subring of R.
(3) Show that if a rational number is an algebraic integer, then it is an integer.
(4) Show that character values are algebraic integers.
(5) Let G be a finite group, and R be the subset of the centre of the group algebra consisting of the linear combinations with integer coefficient of the sums of the conjugacy classes of G.
  (a) Show that R is a commutative subring with unity of the centre of the group algebra.
  (b) Show that all elements of R are integral.