TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 6

Exercise 6.1. Define the character table of a group.

Exercise 6.2. Recall that a character χ is called *linear* if $\chi(1) = 1$.

(1) Prove that a linear character is irreducible.
(2) Exhibit an example of a group G and an irreducible character of G which is not linear.
(3) Prove that the irreducible characters of a finite abelian group are all linear.
(4) Describe the character table of a finite cyclic group.
(5) Show that the linear characters of a finite abelian group G form under pointwise multiplication a group isomorphic to G. (This is called the dual group $G^\star$ of G.)
(6) Let G be a finite abelian group. Show that the map

$$\begin{aligned} G &\to (G^\star)^\star \\ g &\mapsto (\chi \mapsto \chi(g)) \end{aligned}$$

is a natural isomorphism of G with the bidual $(G^\star)^\star$, very much as in the case of vector spaces.

Exercise 6.3. Let G be a finite group, $N \trianglelefteq G$.

(1) If $\rho : G/N \to \mathrm{GL}(V)$ is a representation of G/N, prove that

$$\begin{aligned} \rho' : G &\to \mathrm{GL}(V) \\ g &\mapsto \rho(gN) \end{aligned}$$

is a representation of G.
(2) State and prove the corresponding result for characters.

Exercise 6.4. Let G be a finite group.

(1) If χ is a linear character of G/G', prove that $\chi'(g) = \chi(gG')$ is a linear character of G.
(2) Let χ' be a linear character of G, that is, a morphism $G \to \mathbf{C}^*$.
   (a) Prove that $G' \leq \ker(\chi')$.
   (b) Prove that

$$\begin{aligned} \chi : G/G' &\to \mathbf{C}^* \\ gG' &\mapsto \chi'(g) \end{aligned}$$

   is well defined, and it is a character of G/G'.
   (Hint: χ is well defined, as if $xG' = yG'$, then $x = yc$ for some $c \in G'$, and thus $\chi'(x) = \chi'(y)\chi'(c) = \chi'(y)$, as $c \in G' \leq \ker(\chi')$.)
(3) Prove that there is a one-to-one correspondence between the linear characters of G and those of G/G'.

Exercise 6.5. Determine the character table of S_3.

Exercise 6.6. Let G be a finite group.

(1) Prove that the product $\lambda\chi$ of a character χ of G by a linear character λ of G is a character of G, of the same degree of χ.
(Hint: Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation affording χ. Then

$$\vartheta : G \to \mathrm{GL}(V)$$
$$g \mapsto \lambda(g)\rho(g)$$

affords $\lambda\chi$.)

(2) Prove that χ is irreducible if and only if $\lambda\chi$ is irreducible.
(Hint: This can be done in at least two ways. First method. In the notation of the previous hint, if U is a $\rho(G)$-invariant subspace of V, then $U\vartheta(g) = U\lambda(g)\rho(g) = U\rho(G) = U$, as $\lambda(g) \in \mathbf{C}^*$, and U is a subspace, so H is also $\vartheta(G)$-invariant. The converse is immediate. Second method.

$$\begin{aligned}(\lambda\chi, \lambda\chi) &= \frac{1}{|G|}\sum_{g\in G} \lambda(g)\chi(g)\overline{\lambda(g)\chi(g)}\\ &= \frac{1}{|G|}\sum_{g\in G} \lambda(g)\chi(g)\lambda(g)^{-1}\overline{\chi(g)}\\ &= \frac{1}{|G|}\sum_{g\in G} \chi(g)\overline{\chi(g)} = (\chi, \chi),\end{aligned}$$

as $\lambda(g)$ is a scalar, and a root of unity.)

Exercise 6.7. Let $\rho_1, \dots, \rho_t$ be the pairwise non-isomorphic irreducible representations, and $\chi_1, \dots, \chi_t$ their characters.

(1) Prove that the components ρ_i^{jk} are (up to suitable constants) an orthonormal basis with respect to the natural inner product on the space of functions $G \to \mathbf{C}$.

(2) Prove that the χ_i are an orthonormal basis with respect to a certain inner product on the space of class functions $G \to \mathbf{C}$, i.e. of the functions which are constant on conjugacy classes.