

**TRENTO, 2022/23**  
**ADVANCED GROUP THEORY**  
**EXERCISE SHEET # 5**

*Exercise 5.1.*

- (1) Define the character of a representation.
- (2) Show that the character depends only on the isomorphism class of a representation, that is, isomorphic representations have the same character.
- (3) Show that characters are class functions, that is, if  $\chi$  is a character of the finite group  $G$ , then

$$\chi(x^{-1}gx) = \chi(g)$$

for all  $x, g \in G$ .

*Exercise 5.2.* Show that the characters of the irreducible representations of a finite group  $G$  are orthonormal in the space of functions  $G \rightarrow \mathbf{C}$  with inner product

$$(a, b) = \frac{1}{|G|} \sum_{g \in G} a(g) \overline{b(g)}.$$

*Exercise 5.3.* Let  $V$  be a  $\mathbf{C}[G]$ -module, and let

$$V \cong n_1 V_1 \oplus \cdots \oplus n_l V_l,$$

where the  $V_i$  are irreducible modules. Let  $\chi$  be the character of  $V$ , and  $\chi_i$  be the character of  $V_i$ .

- (1) Show that  $(\chi, \chi_i) = n_i$ .
- (2) Show that  $V$  determines uniquely the  $n_i$ .
- (3) Show that  $\chi$  determines  $V$  up to isomorphism.

*Exercise 5.4.* Let  $\psi$  be the character of the right regular representation.

- (1) Show that for  $g \in G$

$$\psi(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$

- (2) If  $\chi$  is an irreducible character, show that

$$(\psi, \chi) = \chi(1).$$

- (3) Show that

$$\psi = \sum_{\chi \in \text{Irr}(G)} \chi(1) \chi,$$

where  $\text{Irr}(G)$  is the set of the irreducible characters of  $G$ .

- (4) Show that

$$|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2.$$

*Exercise 5.5.*

- (1) Show that there is an isomorphism of rings/algebras

$$\mathbf{C}[G] \cong \sum_{\chi \in \text{Irr}(G)} M_{\chi(1)}(\mathbf{C}),$$

where  $M_n(\mathbf{C})$  is the algebra of  $n \times n$  matrices.

- (2) Show that the number of irreducible characters of the group  $G$  equals the number of conjugacy classes of  $G$ .