TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 4

Exercise 4.1. Show that every finite-dimensional representation of a finite group G is completely reducible, that is, the direct sum of irreducible ones.

Exercise 4.2. Let $(\cdot, \cdot)$ be an inner product on the finite dimensional $\mathbf{C}$-vector space V.

(1) Show that if U is a subspace of V, then $V = U \oplus U^{\perp}$.
(2) Suppose now $(\cdot, \cdot)$ is the standard inner product, and let $A \in End_F(V)$.
    (a) Show that $A^* = \overline{A}^t$.
    (b) Show that A is unitary if and only if $A^* = A^{-1}$.

Exercise 4.3. Let G be a finite group

(1) Let V a finite-dimensional $\mathbf{C}$-vector space, and $\rho : G \to \mathrm{GL}(V)$ a representation. Show that there is an inner product on V so that all elements of $\rho(G)$ are unitary.
(2) Give a proof of Maschke's Theorem using inner products.

Exercise 4.4.

(1) State and prove Schur's Lemma.
(2) Show that if $\rho : G \to \mathrm{GL}(V)$ is irreducible, and $f : V \to V$ is an isomorphism of representations, then $vf = \lambda v$ for some $\lambda \in \mathbf{C}^*$.
(3) State and prove the orthogonality relations for the components of the irreducible representations.