TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 3

Exercise 3.1.
Let $\rho_i : G \to \mathrm{GL}(V_i)$ be two representations of the finite group G.

(1) Define a morphism of representations.
(2) If $f : V_1 \to V_2$ is a morphism of the representations ρ_1, ρ_2, show that
  (a) the kernel $\ker(f)$ of f is a $\rho_1(G)$-invariant subspace of V_1, and
  (b) the image $V_1 f$ of f is a $\rho_2(G)$-invariant subspace of V_2.
(3) Let $f : V_1 \to V_2$ be a linear map. Prove that

$$\varphi = \sum_{g \in G} \rho_1(g^{-1}) f \rho_2(g) : V_1 \to V_2$$

is a morphism of representations.

Exercise 3.2.

(1) Give the definition of an irreducible representation, and of a direct sum of representations.
(2) State and prove Maschke's theorem

Exercise 3.3. Let F be the field of two elements, and $V = F^2 = \{ [x, y] : x, y \in F \}$.
Let $G = \langle g \rangle$ be the cyclic group of order 2.

(1) Show that

$$\rho(g) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

defines a representation $\rho : G \to \mathrm{GL}(V)$.
(2) Show that $U = \langle [1, 0] \rangle = \{ [0, 0], [1, 0] \}$ is a $\rho(G)$-invariant subspace.
(3) Show that there are no other $\rho(G)$-invariant one-dimensional subspaces.