TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 2

Exercise 2.1.

(1) Show how to obtain a representation of a finite group G out of an action of G on a finite set.
(2) Give the definition of a $\rho(G)$-invariant subspace, of a subrepresentation, and of a direct sum of representations.
(3) Consider the representation of the cyclic group of order 2 derived by its natural action on $\{1, 2\}$, and show it decomposes as the sum of two 1-dimensional representations.
(4) Consider the representation of the cyclic group of order 3 derived by its natural action on $\{1, 2, 3\}$, and show it decomposes as the sum of three 1-dimensional representations.
(5) Consider the representation of the symmetric group S_3 derived by its natural action on $\{1, 2, 3\}$, and show it decomposes as the sum of 1-dimensional representation and a 2-dimensional representation.

Exercise 2.2. Let $G = \langle (12) \rangle$, a cyclic group of order 2, act naturally on $\Omega = \{1, 2\}$.

(1) Define the representation $\rho : G \to \mathrm{GL}(V)$ naturally associated to it, where V is a $\mathbf{C}$-vector space of dimension 2, with basis v_1, v_2.
(2) Write the matrix of $\rho((12))$ with respect to the basis v_1, v_2.
(3) Let $w_1 = v_1 + v_2$ and $w_2 = v_1 - v_2$. Show that
   (a) w_1, w_2 is a basis of V.
   (b) $W_1 = \langle w_1 \rangle$ and $W_2 = \langle w_2 \rangle$ are $\rho(G)$-invariant subspaces.
   (c) Write the matrix of $\rho((12))$ with respect to the basis w_1, w_2.
   (d) Show that $V = W_1 \oplus W_2$ is a direct sum of representations.

Exercise 2.3. Let $G = \langle (123) \rangle$, a cyclic group of order 3, act naturally on $\Omega = \{1, 2, 3\}$.

(1) Define the representation $\rho : G \to \mathrm{GL}(V)$ naturally associated to it, where V is a $\mathbf{C}$-vector space of dimension 3, with basis v_1, v_2, v_3.
(2) Write the matrix of $\rho((123))$ with respect to the basis v_1, v_2, v_3.
(3) Let

$$\begin{cases} w_0 = v_1 + v_2 + v_3 \\ w_1 = v_1 + \omega v_2 + \omega^2 v_3 \\ w_2 = v_1 + \omega^2 v_2 + \omega v_3 \end{cases}$$

   Show that
   (a) w_0, w_1, w_2 is a basis of V.
   (b) $W_0 = \langle w_0 \rangle$, $W_1 = \langle w_1 \rangle$ and $W_2 = \langle w_2 \rangle$ are $\rho(G)$-invariant subspaces.
   (c) Write the matrix of $\rho((123))$ with respect to the basis w_1, w_2, w_3.
   (d) Show that $V = W_1 \oplus W_2 \oplus W_3$ is a direct sum of representations.

Exercise 2.4. Let G be a finite group, $\mathbf{C}[G]$ the set of functions $G \to \mathbf{C}$.

(1) Show that $\mathbf{C}[G]$ becomes a $\mathbf{C}$-vector space with the operations by component:
$$(\lambda a + \mu b)(x) = \lambda a(x) + \mu b(x),$$
for $\lambda, \mu \in \mathbf{C}$, $a, b \in \mathbf{C}[G]$, $x \in G$.

(2) Define on $\mathbf{C}[G]$ the convolution product to be
$$(a * b)(g) = \sum_{xy=g} a(x)b(y).$$
Show that this is associative, and that with these operations $\mathbf{C}[G]$ becomes a ring.

(3) Define, for $g \in G$, the element of $\mathbf{C}[G]$
$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{if } x \neq g. \end{cases}$$
Show that $\delta_g * \delta_h = \delta_{gh}$ for all $g, h \in G$, so that the map $g \mapsto \delta_g$ is a group isomorphism $G \to \{ \delta_g : g \in G \}$.

Notice

The next two exercises are for reference. The universal property of the group algebra will play a role later, and the equivalence of Exercise 2.6 (if not the details of the proof) is essential

Exercise 2.5.

(1) Give the definition of an algebra over a field.
(2) Show that the $n \times n$ matrices over a field F form an F-algebra.
(3) Show that the group algebra $\mathbf{C}[G]$ is indeed an algebra.
(4) State and prove the universal property of the group algebra.

Exercise 2.6.

(1) Let R be a unital ring, M an abelian group. Define what it means for M to have the structure of a right or left R-module.
(2) Let G be finite group, and V a finite-dimensional vector space over $\mathbf{C}$.
   (a) Show that a representation $\rho : G \to \mathrm{GL}(V)$ yields a structure of a $\mathbf{C}[G]$-module on V.
   (b) Show that if $(G, \cdot, 1)$ is a group, $(M, \cdot, 1)$ is a monoid, and $\varphi : G \to M$ is a morphism of monoids (meaning $\varphi(gh) = \varphi(g)\varphi(h)$ for $g, h \in G$, and $\varphi(1) = 1$), then the image $\varphi(G)$ of φ is a group under the operation "$\cdot$" of M.
   (c) Given a structure of a $\mathbf{C}[G]$-module on V, show that this yields a representation $\rho : G \to \mathrm{GL}(V)$.