TRENTO, 2022/23
ADVANCED GROUP THEORY
EXERCISE SHEET # 1

Exercise 1.1.

(1) Give the definition of a (linear) representation ρ of a finite group G on a finite dimensional $\mathbf{C}$-vector space V.

(In the lectures, I have only talked of a representation as a morphism $\rho : G \to \mathrm{GL}(n, \mathbf{C})$, where $\mathrm{GL}(n, \mathbf{C})$ is the group of $n \times n$ invertible matrices with complex coefficients, but one can also define a representation as a morphism $\rho : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of invertible linear maps on V. Once you choose a basis of V, you get the first definition.)

(2) Show that if $\rho : G \to \mathrm{GL}(V)$ is a representation of a finite group G of order k, then there is a change of basis on V such that

$$\rho(g) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

where the λ_i are k-th roots of unity.

Exercise 1.2 (This will be useful later). Let the group G act (on the right) on the set Ω.

Let Δ be another set, and let Δ^Ω be the set of functions $\Omega \to \Delta$.

Show that setting, for $\varphi \in \Delta^\Omega$ and $g \in G$,

$$\varphi^g(x) = \varphi(x^{g^{-1}}), \quad \text{for } x \in \Omega$$

defines a (right) action of G on Δ^Ω.

(This is slightly counterintuitive. We have to prove that for $\varphi \in \Delta^\Omega$ and $g, h \in G$ we have

$$\varphi^{gh} = (\varphi^g)^h.$$

Note that if $\psi \in \Delta^\Omega$, then

$$\psi^h(x) = \psi(x^{h^{-1}}).$$

Setting $\psi = \varphi^g$, we obtain, for $x \in \Omega$,

$$(\varphi^g)^h(x) = \varphi^g(x^{h^{-1}}) = \varphi((x^{h^{-1}})^{g^{-1}}) = \varphi(x^{h^{-1}g^{-1}}) = \varphi(x^{(gh)^{-1}}) = \varphi^{gh}(x),$$

as desired.)

Exercise 1.3.

(1) State von Dyck's theorem.

(2) Let

$$S_3 = \left\langle a_1, a_2 : a_1^3, a_2^2, (a_1 a_2)^2 \right\rangle.$$

Let ω be a primitive third root of unity. Let

$$h_1 = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \qquad h_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

. Show that there is a representation $\rho : S_3 \to \mathrm{GL}(2, \mathbf{C})$ such that

$$\rho(a_i) = h_i, \quad \text{for } i = 1, 2.$$

(3) Show that the minimal polynomial of h_1 is $x^2 + x + 1$, and the minimal polynomial of h_2 is $x^2 - 1$.
(4) Find a basis with respect to which h_2 is diagonal, and find the form of h_1 with respect to such a basis.
(5) Show that $h_1 h_2 \neq h_2 h_1$, so that the image $\rho(S_3)$ of ρ is non-abelian.
(6) List the elements of $\rho(S_3)$.
(7) Show that ρ is injective, so that $\rho(S_3) \cong S_3$.

Exercise 1.4 (To be completed next week).
Let $V = \mathbf{C}^n$ be a $\mathbf{C}$-vector space of dimension n.
Let $\mathcal{M}$ be a set of $n \times n$ matrices, such that each $M \in \mathcal{M}$ is diagonalisable.
Show that the following are equivalent.

(1) There is a basis of V with respect to which all $M \in \mathcal{M}$ are diagonal, and
(2) $MN = NM$ for all $M, N\mathcal{M}$.