DIARY OF THE COURSE ADVANCED GROUP THEORY

A.A. 2020/21

INSTRUCTOR: ANDREA CARANTI

Lecture 1. Monday 22 February 2021 (1 hour)

Series and normal series in a group. Right operator groups: Ω-groups. Ω-subgroups and Ω-series. Ω-composition series: composition series and principal series.

Lecture 2. Tuesday 23 February 2021 (2 hours)

The factors of a composition series are simple groups.

The factors H_{i+1}/H_i of a principal series are minimal normal subgroups of G/H_i.

A minimal normal subgroup of a finite group is a characteristically simple groups.

An abelian minimal normal subgroup is an elementary abelian p-group, for some prime p.

Elementary abelian groups and vector spaces.

A non-abelian minimal normal subgroup is the direct product of isomorphic copies of the same simple group.

Lemma: $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$.

The minimal normal subgroups of a direct product of isomorphic copies of the same *nonabelian* simple group is characteristically simple are the factors.

Lecture 3. Monday 1 March 2021 (1 hour)

Action of the symmetric group on S^n.

Conversely, a direct product of isomorphic copies of the same simple group is characteristically simple.

Uniqueness of factors of an Ω-composition series.

Commutators, derived/commutator subgroup.

Lecture 4. Tuesday 2 March 2021 (2 hours)

Some commutator calculus.

Properties of the derived subgroup. The derived subgroup is the smallest normal subgroup with an abelian quotient.

Soluble groups: equivalent statements.

Properties of soluble groups with respect to subgroups, quotient groups and extensions.

Trento, 2020/21.

In a soluble group, the quotient of a composition (resp. principal) series are cyclic of prime order (resp. elementary abelian p-groups).

Lecture 5. Monday 8 March 2021 (1 hour)

Group actions in two ways.
Representations of groups. Eigenvalues. From group actions to representations.
Inner products: a representation can be taken to be unitary.
The regular representation of a cyclic group of order 2.
Invariant spaces and subrepresentations.

Lecture 6. Tuesday 9 March 2021 (2 hours)

The group algebra $\mathbf{C}[G]$.
Representations of a group G over a vector space V are equivalent to structures of $\mathbf{C}[G]$-module on V.
Examples. Maschke's Theorem (statement and beginning of proof).

Lecture 7. Monday 15 March 2021 (1 hour)

Maschke's Theorem (end of both proofs).
Morphisms of representation.
Irreducible representations/simple modules.
Direct sum of representations.
Corollary of Maschke's Theorem: every representation is the direct sum of irreducible ones.

Lecture 8. Tuesday 16 March 2021 (2 hours)

Again on the direct sum of representations.
Isomorphism of representations.
Schur's Lemma and its Corollary.
How to make a morphism of representations out of any linear map.
Adjoint matrices, unitary matrices.
Orthogonality of the components of the irreducible representations (first part).

Lecture 9. Monday 22 March 2021 (1 hour)

Orthogonality of the components of the irreducible representations (second part).
Characters. Characters depend only on the isomorphism class of a representation.
Irreducible characters are orthonormal (first part).

Lecture 10. Tuesday 23 March 2021 (2 hours)

Irreducible characters are orthonormal (second part).

The right regular representation and its character.

A character uniquely determines a representation. In particular, a character χ is the character of an irreducible representation if and only if $(\chi, \chi) = 1$. (Hence we can use the term *irreducible* character.)

Examples of the right regular representation and its character.

An irreducible character χ appears in the regular representation $\chi(1)$ times.

$$\rho = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi,$$

hence

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

First part of: the isomorphism

$$\mathbf{C}[G] \cong \bigoplus_{\chi \in \mathrm{Irr}(G)} M_{\chi(1)}(\mathbf{C}).$$

Lecture 11. Monday 29 March 2021 (1 hour)

Second part of: the isomorphism

$$\mathbf{C}[G] \cong \bigoplus_{\chi \in \mathrm{Irr}(G)} M_{\chi(1)}(\mathbf{C}).$$

Computing the centres of the two algebras, ones gets that the number of irreducible characters of a finite group G equals the number of its conjugacy classes.

Lecture 12. Tuesday 30 March 2021 (2 hours)

Characters are class functions, and they are a basis of the space of class functions.

The irreducible characters χ of a finite abelian group G are all linear, i.e. $\chi(1) = 1$.

Another proof, via commuting matrices, that the irreducicble characters of a finite abelian group are linear.

Linear characters of finite cyclic groups.

The product of a linear character and a(n irreducible) character is a(n irreducible) character.

The dual group.

Lecture 13. Monday 12 April 2021 (1 hour)

Recap.

Lecture 14. Tuesday 13 April 2021 (2 hours)

Recap.
Linear characters of finite abelian groups.
Representations from a quotient group to the full group.
The linear characters of a finite group G correspond to the linear characters of G/G'.
The character table of S_3.
(As a consequence of the product of an irreducible character and of a linear character, the zero in the non-linear character of S_3.)

Lecture 15. Monday 19 April 2021 (1 hour)

Permutation representations and characters. Number of orbits as the average of fixed points.

Lecture 16. Tuesday 20 April 2021 (2 hours)

Doubly transitive groups. More on the characters of S_3.
Irreducible characters of A_4 and S_4. Decomposition $S_4 = V S_3$.

Lecture 17. Monday 26 April 2021 (1 hour)

The image of a central element under the isomorphism $\mathbf{C}[G] \to \bigoplus_{\chi \in \mathrm{Irr}(G)} \mathrm{End}(V_\chi)$.
Three equivalent conditions for an element of a commutative ring of characteristic zero to be integral.

Lecture 18. Tuesday 27 April 2021 (2 hours)

Integral elements and algebraic integers. A rational number which is algebraic is an integer.
The degree of an irreducible character divides the order of the group.
Lemma on norms in a Galois extension with abelian Galois group.

Lecture 19. Monday 3 May 2021 (1 hour)

A counterexample to the Lemma when the Galois group is not abelian.
A nonlinear, irreducible character always takes the value zero.

Lecture 20. Tuesday 4 May 2021 (2 hours)

Lemma on sums of complex numbers of norm 1, and sums of roots of unity.
Corollary. If ρ affords χ, and $g \in G$, then $\chi(g) = \chi(1)$ if and only if $g \in \ker(\rho)$.
The character table is a non-singular matrix. The extended character table. The other orthogonality relations. Size of a centraliser when passing to a quotient group: a proof with characters.

Lecture 21. Monday 10 May 2021 (1 hour)

Size of a centraliser when passing to a quotient group: a character-free proof.

Products of conjugacy classes and structure constants of the centre of the group algebra — the statement.

Lecture 22. Tuesday 11 May 2021 (2 hours)

Products of conjugacy classes and structure constants of the centre of the group algebra — the proof and an example.

Tensor products over non-commutative rings. Bimodules.

Lecture 23. Monday 17 May 2021 (1 hour)

Induced representations.

Lecture 24. Tuesday 18 May 2021 (2 hours)

Induced characters. Frobenius reciprocity.

Burnside's $p^a q^b$ theorem.

Lecture 25. Monday 31 May 2021 (1 hour)

Frobenius groups.

Lecture 26. Tuesday 1 June 2021 (2 hours)

Existence of the Frobenius kernel.

Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, 38123 Trento

Email address: andrea.caranti@unitn.it

URL: https://caranti.maths.unitn.it