DIARY OF THE COURSE ADVANCED GROUP THEORY

A.A. 2022/23

INSTRUCTOR: ANDREA CARANTI

Please note: The description of lectures not yet given is meant as planning.

Lecture 1. Monday 12 September 2022 (2 hours)

Group actions in two ways.
Representations of groups. Eigenvalues. Diagonalisation.
A representation of the cyclic group of order 2.
Presentation of $S_3 = D_3$.

Lecture 2. Wednesday 14 September 2022 (1 hour)

An example: a (faithful) representation of $S_3 = D_3$. Von Dick's theorem.
Simultaneous diagonalisation and commuting matrices.

Lecture 3. Monday 19 September 2022 (2 hours)

Simultaneous diagonalisation and commuting matrices. (Conclusion, I have begun the "not all scalar" case.)
Representations of abelian groups. Invariant spaces and subrepresentations. Irreducible representations. Direct sum of representations.
From group actions to representations: the regular representation. The natural representation of S_3 on $\{ 1, 2, 3 \}$ and how it decomposes.
The group algebra $\mathbf{C}[G]$ (introduction).

Lecture 4. Wednesday 21 September 2022 (1 hour)

The group algebra $\mathbf{C}[G]$ and its universal property.
Representations of a group G over a vector space V are equivalent to structures of $\mathbf{C}[G]$-module on V. (To be completed.)

Lecture 5. Wednesday 28 September 2022 (2 hours)

Representations of a group G over a vector space V are equivalent to structures of $\mathbf{C}[G]$-module on V.
Irreducible representations/simple modules. Direct sum of representations/modules.
Morphisms and isosomorphism of representations. How to make a morphism of representations out of any linear map.
Maschke's Theorem.

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Lecture 6. Monday 3 October 2022 (2 hours)

Corollary of Maschke's Theorem: every representation is the direct sum of irreducible ones.

Inner products: adjoint matrices, unitary matrices, a representation can be taken to be unitary.

Second proof of Maschke's Theorem.

Lecture 7. Wednesday 5 October 2022 (1 hour)

Schur's Lemma and its Corollary.

Orthogonality of the components of the irreducible representations (first part).

Lecture 8. Monday 10 October 2022 (2 hours)

Orthogonality of the components of the irreducible representations (second part).

Characters.

Characters are class functions.

Characters depend only on the isomorphism class of a representation.

Irreducible characters are orthonormal.

A character uniquely determines a representation. In particular, a character χ is the character of an irreducible representation if and only if $(\chi, \chi) = 1$. (Hence we can use the term *irreducible* character.)

Lecture 9. Wednesday 12 October 2022 (2 hours)

Irreducible representations and irreducible characters.

The right regular representation and its character.

Examples of the right regular representation and its character.

An irreducible character χ appears in the regular representation $\chi(1)$ times.

$$\rho = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi,$$

hence

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

The isomorphism

$$\mathbf{C}[G] \cong \bigoplus_{\chi \in \mathrm{Irr}(G)} M_{\chi(1)}(\mathbf{C}).$$

Computing the centres of the two algebras, ones gets that the number of irreducible characters of a finite group G equals the number of its conjugacy classes.

The irreducible characters χ of a finite abelian group G are all linear, i.e. $\chi(1) = 1$.

Lecture 10. Monday 17 October 2022 (2 hours)

All abelian groups of the same order have isomorphisc group algebras.

Linear characters are morphisms $G \to \mathbf{C}^\star$. The irreducible (linear) characters of finite cyclic groups.

The dual group: linear characters of finite abelian groups.

Representations from a quotient group to the full group.

The derived subgroup $G' = [G, G]$ of a group G.

The linear characters of a finite group G correspond to the linear characters of G/G'.

Character tables.

The character table of S_3.

The product of a linear character and a(n irreducible) character is a(n irreducible) character. As a consequence the zero in the non-linear character of S_3; mention the general result.

Lecture 11. Wednesday 19 October 2022 (1 hour)

The ρ_i^{jk} are up to constants an orthonormal base of the space $\mathbf{C}^G$. The irreducible characters are a basis of the space of class functions.

The product of two character is a character (beginning).

Lecture 12. Monday 24 October 2022 (2 hours)

The product of two character is a character (end).

The character ring: S_3 as an example.

Permutation representations and characters. Number of orbits as the average of fixed points.

Doubly transitive groups. More on the characters of S_3.

Irreducible characters of A_4.

Lecture 13. Wednesday 26 October 2022 (2 hours)

The irreducible characters of S_4. Decomposition $S_4 = V S_3$.

Integral elements and algebraic integers. A rational number which is algebraic is an integer.

Three equivalent conditions for an element of a commutative ring of characteristic zero to be integral.

The subring of the centre of the group algebra consisting of the linear combinations with integer coefficients of the sums of the conjugacy classes.

Lecture 14. Wednesday 2 November 2022 (1 hour)

The image of a central element under the isomorphism $\mathbf{C}[G] \to \bigoplus_{\chi \in \mathrm{Irr}(G)} \mathrm{End}(V_\chi)$.

The degree of an irreducible character divides the order of the group.

Lemma on norms in a Galois extension with abelian Galois group.

Lemma on sums of complex numbers of norm 1, and sums of roots of unity.

Lecture 15. Monday 7 November 2022 (2 hours)

If ρ affords χ, and $g \in G$, then $\chi(g) = \chi(1)$ if and only if $g \in \ker(\rho)$.

A nonlinear, irreducible character always takes the value zero.

The character table is a non-singular matrix. The extended character table. The other orthogonality relations. Size of a centraliser when passing to a quotient group: a proof with characters.

Size of a centraliser when passing to a quotient group: a character-free proof.

Induced characters.

Lecture 16. Wednesday 9 November 2022 (2 hours)

Frobenius reciprocity.

Balanced products. Tensor products of modules over non-commutative rings. Bimodules. Induced representations.

Series and normal series in a group. Right operator groups: Ω-groups. Ω-subgroups.

Subgroups ($\Omega = \emptyset$), normal subgroups ($\Omega = \mathrm{Inn}(G)$), characteristic subgroups ($\Omega = \mathrm{Aut}(G)$), fully invariant subgroups ($\Omega = \mathrm{Aut}(G)$).

Examples to see that the four concepts are distinct.

(1) A subgroup which is not normal.

Elementary abelian p-groups as vector spaces over $\mathrm{GF}(p)$.

Lecture 17. Monday 14 November 2022 (2 hours)

Elementary abelian p-groups as vector spaces over $\mathrm{GF}(p)$.

(2) A normal subgroup which is not characteristic.
(3) A characteristic subgroup which is not fully invariant.

We have:

(1) a normal subgroup H of a normal subgroup N of the group G is not necessarily normal in G, but
(2) a characteristic subgroup H of a normal subgroup N of the group G is normal in G, and
(3) a characteristic subgroup H of a characteristic subgroup N of the group G is characteristic in G.

Ω-sequences and Ω-series. Ω-composition series: composition series (when $\Omega = \emptyset$) and principal series (when $\Omega = \mathrm{Inn}(G)$).

Uniqueness of factors of an Ω-composition series.

Lecture 18. Wednesday 16 November 2022 (1 hour)

Composition factors are simple groups.

The factors H_{i+1}/H_i of a principal series are minimal normal subgroups of G/H_i, and thus characteristically simple groups.

Lemma: $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. Characteristically simple groups are direct products of isomorphic copies of the same simple group.

Conversely: An elementary abelian p-group is characteristically simple.

Lecture 19. Monday 21 November 2022 (2 hours)

More converse:

(1) The minimal normal subgroups of a direct product S^n of n isomorphic copies of the same *nonabelian* simple group S are the factors.
(2) Action of the symmetric group on S^n.
(3) A direct product of isomorphic copies of the same simple group is characteristically simple.

Properties of the derived subgroup. The derived subgroup is the smallest normal subgroup with an abelian quotient.

Soluble groups: equivalent statements.

Properties of soluble groups with respect to subgroups, quotient groups and extensions.

In a soluble group, the quotient of a composition (resp. principal) series are cyclic of prime order (resp. elementary abelian p-groups).

Statement of Burnside's theorem about the solubility of groups of order $p^a q^b$: first part of the Lemma.

Lecture 20. Wednesday 23 November 2022 (2 hours)

Second part of the Lemma, and proof of Burnside's theorem about the solubility of groups of order$p^a q^b$.

Frobenius groups: equivalent statements. Examples. Statement of Frobenius' theorem on the existence of the Frobenius kernel.

Lecture 21. Monday 28 November 2022 (2 hours)

Proof of Frobenius' theorem on the existence of the Frobenius kernel.

The character table of S_5.

Lecture 22. Wednesday 30 November 2022 (2 hours)

The character table of S_5 and A_5: Galois-conjugate characters.

Permutations of rows and columns of the character table. The only real character of a group of odd order is the trivial character.

Clifford theory (beginning).

Lecture 23. Monday 5 December 2022 (2 hours)

Clifford theory.

An application to the nonabelian group of order 21.

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