TRENTO, 2020/21
ADVANCED GROUP THEORY
EXERCISE SHEET # 10

Exercise 10.1 (Just for curiosity). Give an example of a Galois extension $E/\mathbf{Q}$, with $E \subseteq \mathbf{C}$ such that there is $\alpha \in E$ and $g \in \mathrm{Gal}(E/\mathbf{Q})$ for which

$$| \alpha^g | \neq | \alpha |^g .$$

Exercise 10.2. Prove that if G is a group, $\chi \in \mathrm{Irr}(G)$, then the following are equivalent

(1) $\chi(g) \neq 0$ for all $g \in G$, and
(2) χ is linear, that, $\chi(1) = 1$.

Exercise 10.3. Let $z_1, \dots, z_n \in \mathbf{C}$ have all absolute value 1.
Prove that if $| z_1 + \dots + z_n | = n$, then $z_1 = \dots = z_n$.

Exercise 10.4. Let $\omega_1, \dots, \omega_n \in \mathbf{C}$ be roots of unity.
Prove that if

$$\alpha = \frac{\omega_1 + \dots + \omega_n}{n}$$

is an algebraic integer, then either $\alpha = 0$, or $\omega_1 = \dots = \omega_n = \alpha$.

Exercise 10.5. Let χ be the character of the representation ρ of teh finite group G.
Prove that

$$\ker(\rho) = \{ g \in G : \chi(g) = \chi(1) \} .$$

Exercise 10.6. Prove that the character table of a finite group is a non-singular matrix.

Exercise 10.7. Prove the “other” ortogonality relations, that is, if G is a finite group, and $x, y \in G$, then

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(x)\chi(y^{-1}) = \begin{cases} | C_G(x) | & \text{if } x \text{ and } y \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 10.8. Show using characters that if G is a finite group, $N \trianglelefteq G$ and $x \in G$, then

$$| C_G(x) | \geq \left| C_{G/N}(xN) \right| .$$