TRENTO, 2020/21
ADVANCED GROUP THEORY
EXERCISE SHEET # 9

Exercise 9.1. Let $f : G \to \mathbf{C}$ be a class function, so that $\alpha = \sum_{g \in G} f(g) g \in Z(\mathbf{C}[G])$.

Let $\rho : G \to \mathrm{GL}(V_\chi)$ be an irreducible representation with character χ. Let

$$\begin{aligned} \varphi : \mathbf{C}[G] &\to \mathrm{End}(V_\chi) \\ \sum_{g \in G} a(g) g &\mapsto \sum_{g \in G} a(g) \rho(g) \end{aligned}$$

be the natural epimorphism of algebras.

Show that $\varphi(\alpha)$ is a scalar matrix which has on the diagonal the number

$$\frac{|\, G \,|}{\chi(1)} \cdot (f, \overline{\chi}).$$

Exercise 9.2 (Optional, it plays a role in the next exercise).

(1) Prove Dedekind's Identity: Let A, B, C be submodules of a module M, with $A \supseteq B$. Then

$$A \cap (B + C) = B + (A \cap C).$$

(2) Show that if A, B, C are submodules of a module M, the identity

$$A \cap (B + C) = (A \cap B) + (A \cap C)$$

does not hold in general.
(HINT: Take M to be a vector space of dimension 2 over your favourite field, and A, B, C be three distinct subspaces of dimension 1.)

Exercise 9.3 (Just for reference). Let A be a commutative, unital Noetherian ring. Show that each finitely generated A-module is Noetherian.

Exercise 9.4. Let R be a commutative, unital ring of characteristic zero, so that $\mathbf{Z}$ is a subring with unity of R.

(1) Show that for $\alpha \in R$, the following are equivalent:
   (a) there exists $n \geq 1$ and $a_1, \dots, a_n \in \mathbf{Z}$ such that

   $$\alpha^n + a_1 \alpha^{n-1} + \cdots + a_n = 0.$$

   (b) The subring

   $$\mathbf{Z}[\alpha] = \left\{ a_0 + a_1 \alpha + \cdots + a_k \alpha^k : k \in \mathbf{N}, a_i \in \mathbf{Z} \right\}$$

   of R is finitely generated as an abelian group, that is, as a $\mathbf{Z}$-module.
   (c) The subring $\mathbf{Z}[\alpha]$ of R is *contained* in a finitely generated $\mathbf{Z}$-submodule of R.

An element α satisfying these conditions is said to be *integral*. If $R = \mathbf{C}$, then α is said to be an *algebraic integer*.

(2) Show that the integral elements of R form a subring of R.

(3) Show that if a rational number is an algebraic integer, then it is an integer.
(4) Show that character values are algebraic integers.
(5) Show that if $\chi \in \operatorname{Irr}(G)$, and $f : G \to \mathbf{C}$ is a class function, whose values are algebraic integers, then

$$\frac{1}{\chi(1)} \sum_{g \in G} f(g)\chi(g)$$

is an algebraic integer.
(6) Show that if $\chi \in \operatorname{Irr}(G)$, then $\chi(1)$ divides the order of G.