TRENTO, 2020/21
ADVANCED GROUP THEORY
EXERCISE SHEET # 8

Exercise 8.1. Let G be a finite group, $\rho : G \to \mathrm{GL}(V)$ a representation of G, and λ a linear character of G.

(1) Show that $\lambda\rho$, defined as $(\lambda\rho)(g) = \lambda(g)\rho(g)$ is a representation $G \to \mathrm{GL}(V)$.
(2) If the character of ρ is χ, show that the character of $\lambda\rho$ is $\lambda\chi$.
(3) Show that ρ is irreducible if and only if $\lambda\rho$ is. Do it in two ways
   (a) Using the fact that a character χ is irreducible iff $(\chi, \chi) = 1$.
   (b) Using directly the definition of an irreducible representation.
(4) Use the above to show that for the non-linear character χ of S_3 we must have $\chi((12)) = 0$.

Exercise 8.2. Let G be finite group acting on the finite set Ω.

(1) Define the associated permutation representation ρ and its character χ.
(2) Show that $\chi(g) = F(g) = \{ \alpha \in \Omega : \alpha^g = \alpha \}$ is the number of fixed point of g.
(3) Show that the numer of orbits of G on Ω is given by

$$\frac{1}{|G|} \sum_{g \in G} F(g) = (1, \chi),$$

   where 1 denotes the trivial character.
(4) Show that G acts transitively on Ω (i.e., there is only one orbit) iff $\chi = 1 + \psi$, where ψ is a character such that $(1, \psi) = 0$.
(5) Define what is meant for G to act double transitively on Ω (one also says G acts 2-transitively, or that G is 2-transitive).
(6) Show that G is 2-transitive iff ψ is irreducible.

Exercise 8.3. Let G be a finite group acting on the finite set Ω. In general, one says that G acts k-transitively on Ω if for any distinct $\alpha_1, \dots, \alpha_k \in \Omega$, and any distinct $\beta_1, \dots, \beta_k \in \Omega$, there is $g \in G$ such that $\alpha_i^g = \beta_i$ for all i.

(1) Prove that to show that G is k-transitive, it is enough to show that *for a fixed set of distinct* $\gamma_1, \dots, \gamma_k \in \Omega$ and any distinct $\beta_1, \dots, \beta_k \in \Omega$, there is $g \in G$ such that $\gamma_i^g = \beta_i$ for all i.
   (HINT: Given any distinct $\alpha_1, \dots, \alpha_k \in \Omega$ and any distinct $\beta_1, \dots, \beta_k \in \Omega$, by assumption there is a $g \in G$ such that $\gamma_i^g = \alpha_i$ for all i, and an $h \in G$ such that $\gamma_i^g = \beta_i$. Then $\alpha_i^{g^{-1}h} = \gamma_i^h = \beta_i$ for all i.)
(2) Show that if G is k-transitive, for $k > 1$, then it is also $(k-1)$-transitive.
(3) Show that S_n is n-transitive in its natural action on $\{ 1, 2, \dots, n \}$.
(4) Show that A_n is $(n-2)$-transitive, but not $(n-1)$-transitive in its natural action on $\{ 1, 2, \dots, n \}$.
   (HINT: Given distinct $\beta_1, \dots, \beta_{n-2}$, by the above there is $g \in S_n$ such that $i^g = \beta_i$ for all $i = 1, \dots, n-2$. If g is even, we are done. If g is odd, then

$(n-1,n)g$ is even, and $i^{(n-1,n)g} = i^g = \beta_i$ for all $i = 1, \dots, n-2$. This shows that A_n is $(n-2)$-transitive. Now note that the only permutation g such that $i^g = i$ for $i = 1, \dots, n-2$ and $(n-1)^g = n$ is the 2-cycle $(n-1,n)$, which is odd. This shows that A_n is *not* $(n-1)$-transitive.)

Exercise 8.4. Construct the character tables of S_4 and A_4.