TRENTO, 2020/21
ADVANCED GROUP THEORY
EXERCISE SHEET # 5

Exercise 5.1.

(1) Define the character of a representation.
(2) Show that the character depends only on the isomorphism class of a representation, that is, isomorphic representations have the same character.
(3) Show that characters are class functions, that is, if χ is a character of the finite group G, then
$$\chi(x^{-1}gx) = \chi(g)$$
for all $x, g \in G$.

Exercise 5.2. Show that the characters of the irreducible representations of a finite group G are orthonormal in the space of functions $G \to \mathbf{C}$ with inner product
$$(a, b) = \frac{1}{|\, G\,|} \sum_{g \in G} \overline{a(g)} b(g).$$

Exercise 5.3. Let V be a $\mathbf{C}[G]$-module, and let
$$V \cong n_1 V_1 \oplus \cdots \oplus n_l V_l,$$
where the V_i are irreducible modules. Let χ be the character of V, and χ_i be the character of V_i.

(1) Show that $(\chi, \chi_i) = n_i$.
(2) Show that V determines uniquely the n_i.
(3) Show that χ determines V up to isomorphism.

Exercise 5.4. Let ψ be the character of the right regular representation.

(1) Show that for $g \in G$
$$\psi(g) = \begin{cases} |\, G\,| & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$
(2) If χ is an irreducible character, show that
$$(\psi, \chi) = \chi(1).$$
(3) Show that
$$\psi = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi,$$
where $\mathrm{Irr}(G)$ is the set of the irreducible characters of G.
(4) Show that
$$|\, G\,| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

Exercise 5.5 (This will be completed next week). Show that there is an isomorphism of rings/algebras

$$\mathbf{C}[G] \cong \sum_{\chi \in \mathrm{Irr}(G)} M_{\chi(1)}(\mathbf{C}),$$

where $M_n(\mathbf{C})$ is the algebra of $n \times n$ matrices.