TRENTO, 2020/21
ADVANCED GROUP THEORY
EXERCISE SHEET # 4

Exercise 4.1.
(1) Give the definition of an irreducible representation, and of a direct sum of representations.
(2) State and prove Maschke's theorem

Exercise 4.2.
(1) Define a morphism of representations.
(2) If $f : V_1 \to V_2$ is a morphism of the representations ρ_1, ρ_2, show that
  (a) the kernel $\ker(f)$ of f is a $\rho_1(G)$-invariant subspace of V_1, and
  (b) the image $V_1 f$ of f is a $\rho_2(G)$-invariant subspace of V_2.

Exercise 4.3. Let $G = \langle (12) \rangle$, a cyclic group of order 2, act naturally on $\Omega = \{ 1, 2 \}$.
(1) Define the representation $\rho : G \to \mathrm{GL}(V)$ naturally associated to it, where V is a $\mathbf{F}_2$-vector space of dimension 2, with basis v_1, v_2. (Here $\mathbf{F}_2 = \{ 0, 1 \}$ is the field with two elements.)
(2) Write the matrix of $\rho((12))$ with respect to the basis v_1, v_2.
(3) Show that the only $\rho(G)$-invariant subspace of dimension 1 is $\langle v_1 + v_2 \rangle$. (Hint: There are just three subspaces of dimension 1.)
(4) Write the matrix of $\rho((12))$ with respect to the basis $v_1, v_1 + v_2$.

Exercise 4.4. Let $(\cdot, \cdot)$ be an inner product on the finite dimensional $\mathbf{C}$-vector space V.
(1) Show that if U is a subspace of V, then $V = U \oplus U^{\perp}$.
(2) Suppose now $(\cdot, \cdot)$ is the standard inner product, and let $A \in End_F(V)$.
  (a) Show that $A^* = \overline{A}^t$.
  (b) Show that A is unitary if and only if $A^* = A^{-1}$.

Exercise 4.5. Let G be a finite group
(1) Let V a finite-dimensional $\mathbf{C}$-vector space, and $\rho : G \to \mathrm{GL}(V)$ a representation. Show that there is an inner product on V so that all elements of $\rho(G)$ are unitary.
(2) Give a proof of Maschke's Theorem using inner products.
(3) Show that every representation of G is the direct sum of irreducible ones.

Exercise 4.6.
(1) State and prove Schur's Lemma.
(2) Show that if $\rho : G \to \mathrm{GL}(V)$ is irreducible, and $f : V \to V$ is an isomorphism of representations, then $vf = \lambda v$ for some $\lambda \in \mathbf{C}^*$.
(3) State and prove the orthogonality relations for the components of the irreducible representations.