TRENTO, 2020/21
ADVANCED GROUP THEORY
EXERCISE SHEET # 3

Exercise 3.1.

(1) Give the definition of a (linear) representation ρ of a finite group on a dfinite dimensional $\mathbf{C}$-vector space V.
(2) Show how to obtain a representation of a finite group G out of an action of G on a finite set.
(3) Give the definition of a $\rho(G)$-invariant subspace, and of a subrepresentation.

Exercise 3.2. Let $G = \langle\, (12)\, \rangle$, a cyclic group of order 2, act naturally on $\Omega = \{\, 1, 2\, \}$.

(1) Define the representation $\rho : G \to \mathrm{GL}(V)$ naturally associated to it, where V is a $\mathbf{C}$-vector space of dimension 2, with basis v_1, v_2.
(2) Write the matrix of $\rho((12))$ with respect to the basis v_1, v_2.
(3) Let $w_1 = v_1 + v_2$ and $w_2 = v_1 - v_2$. Show that
   (a) w_1, w_2 is a basis of V.
   (b) $W_1 = \langle\, w_1\, \rangle$ and $W_2 = \langle\, w_2\, \rangle$ are $\rho(G)$-invariant subspaces.
   (c) Write the matrix of $\rho((12))$ with respect to the basis w_1, w_2.
   (d) (This requires a definition that will be given next week) Show that $V = W_1 \oplus W_2$ is a direct sum of representations.

Exercise 3.3. Let $G = \langle\, (123)\, \rangle$, a cyclic group of order 3, act naturally on $\Omega = \{\, 1, 2, 3\, \}$.

(1) Define the representation $\rho : G \to \mathrm{GL}(V)$ naturally associated to it, where V is a $\mathbf{C}$-vector space of dimension 3, with basis v_1, v_2, v_3.
(2) Write the matrix of $\rho((123))$ with respect to the basis v_1, v_2, v_3.
(3) Let

$$\begin{cases} w_0 = v_1 + v_2 + v_3 \\ w_1 = v_1 + \omega v_2 + \omega^2 v_3 \\ w_2 = v_1 + \omega^2 v_2 + \omega v_3 \end{cases}$$

   Show that
   (a) w_0, w_1, w_2 is a basis of V.
   (b) $W_0 = \langle\, w_0\, \rangle$, $W_1 = \langle\, w_1\, \rangle$ and $W_2 = \langle\, w_2\, \rangle$ are $\rho(G)$-invariant subspaces.
   (c) Write the matrix of $\rho((123))$ with respect to the basis w_1, w_2, w_3.
   (d) Show that $V = W_1 \oplus W_2 \oplus W_3$ is a direct sum of representations.

Exercise 3.4. Let G be a finite group, $\mathbf{C}[G]$ the set of functions $G \to \mathbf{C}$.

(1) Show that $\mathbf{C}[G]$ becomes a $\mathbf{C}$-vector space with the operations by component:

$$(\lambda a + \mu b)(x) = \lambda a(x) + \mu b(x),$$

   for $\lambda, \mu \in \mathbf{C}$, $a, b \in \mathbf{C}[G]$, $x \in G$.

(2) Define on $\mathbf{C}[G]$ the convolution product to be

$$(a * b)(g) = \sum_{xy=g} a(x)b(y).$$

Show that this is associative, and that with these operations $\mathbf{C}[G]$ becomes a ring.

(3) Define, for $g \in G$, the element of $\mathbf{C}[G]$

$$\delta_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{if } x \neq g. \end{cases}$$

Show that $\delta_g * \delta_h = \delta_{gh}$ for all $g, h \in G$, so that the map $g \mapsto \delta_g$ is a group isomorphism $G \to \{ \delta_g : g \in G \}$.

Notice

The next two exercises are for reference. The universal property of the group algebra will play a role later, and the equivalence of Exercise 3.6 (if not the details of the proof) is essential

Exercise 3.5.

(1) Give the definition of an algebra over a field.
(2) Show that the $n \times n$ matrices over a field F form an F-algebra.
(3) Show that the group algebra $\mathbf{C}[G]$ is indeed an algebra.
(4) State and prove the universal property of the group algebra.

Exercise 3.6.

(1) Let R be a unital ring, M an abelian group. Define what it means for M to have the structure of a right or left R-module.
(2) Let G be finite group, and V a finite-dimensional vector space over $\mathbf{C}$.
   (a) Show that a representation $\rho : G \to \mathrm{GL}(V)$ yields a structure of a $\mathbf{C}[G]$-module on V.
   (b) Show that if $(G, \cdot, 1)$ is a group, $(M, \cdot, 1)$ is a monoid, and $\varphi : G \to M$ is a morphism of monoids (meaning $\varphi(gh) = \varphi(g)\varphi(h)$ for $g, h \in G$, and $\varphi(1) = 1$), then the image $\varphi(G)$ of φ is a group under the operation "$\cdot$" of M.
   (c) Given a structure of a $\mathbf{C}[G]$-module on V, show that this yields a representation $\rho : G \to \mathrm{GL}(V)$.