TRENTO, 2020/21
ADVANCED GROUP THEORY
EXERCISE SHEET # 2

Exercise 2.1. Let S be a set, $n \geq 1$ an integer, $G = S^n$ be the direct product of n copies of S.

(1) Show that the assignment, for $\sigma \in S_n$,
$$(s_1, \dots, s_n)^\sigma = (s_{1\sigma^{-1}}, \dots, s_{n\sigma^{-1}})$$
defines a *right* action of S_n on G.
(HINT: This is slightly tricky: the inverse is needed to make this into a *right* action.)

Let now S be a group, so that G is a direct product.

(2) Show that for each $\sigma \in S_n$ the map σ' given by
$$(s_1, \dots, s_n) \mapsto (s_1, \dots, s_n)^\sigma$$

defines an automorphism of the group G, and actually $\sigma \mapsto \sigma'$ is a morphism $S_n \to \mathrm{Aut}(G)$.

Exercise 2.2.

(1) Show that the direct product
$$N = S^n = S \times \dots \times S$$
of a finite number of isomorphic copies of a finite simple group S is a characteristically simple group.

(2) (Optional) Show that the only minimal normal subgroups of N are the
$$S_i = \{ 1 \} \times \dots \times S \times \dots \times \{ 1 \},$$
where the only component different from $\{ 1 \}$ is the i-th one.

(3) (Optional) Show that a group N as above is a minimal normal subgroup of some group G.

Exercise 2.3. Let G be a finite, elementary abelian p-group of order p^n. Show that $\mathrm{Aut}(G) \cong \mathrm{GL}(n, p)$, the latter being the group of invertible $n \times n$ matrices over the field with p elements.

Exercise 2.4. Let G be a group that admits an Ω-composition series.

Prove that the factors in any two Ω composition series of G are pairwise isomorphic, up to a permutation.

Exercise 2.5. Show that the factors of a composition series of a group do not determine the group uniquely up to isomorphism.
(HINT: This is in the Notes.)

Exercise 2.6. Let $p_1, \dots, p_k$ be distinct primes. Let G be a cyclic group of order $p_1 \cdot \dots \cdot p_k$.

Show that G has $k!$ distinct composition series.

Exercise 2.7. Let G be a group.

(1) Prove that for $a, b \in G$ we have
  (a) $ab = ba$ iff $[a, b] = 1$,
  (b) $a^b = a[a, b]$,
  (c) if H is a group, and $\varphi : G \to H$ is a morphism, then $\varphi([a, b]) = [\varphi(a), \varphi(b)]$.
(2) Prove that the derived subgroup of G is a fully invariant subgroup of G.
(3) Prove that the centre

$$Z(G) = \{ z \in G : gz = zg \text{ for all } g \in G \}$$

of a group is a characteristic subgroup, but in general not a fully invariant one.
(HINT: Consider $G = S_3 \times C_2$.)
(4) Define $G^{(0)} = G$, and $G^{(i+1)} = (G^i)'$ for $i \geq 0$. Prove that each $G^{(i)}$ is normal (actually characteristic, actually fully invariant) in G.
(5) Prove that for $H \leq G$ the following are equivalent
  (a) $H \trianglelefteq G$,
  (b) $[H, G] \leq H$.
(6) Prove that for $H \leq G$ the following are equivalent
  (a) $H \trianglelefteq G$ and G/H is abelian,
  (b) $G' \leq H$.

Exercise 2.8. Let G be a group. Prove that the following are equivalent:
(1) G is soluble, that is $G^{(n)} = \{ 1 \}$ for some n;
(2) there is a normal series $G = G_0 \geq G_1 \geq \cdots \geq G_m = \{ 1 \}$ with G_i/G_{i+1} abelian for all i;
(3) there is a series $G = G_0 \geq G_1 \geq \cdots \geq G_m = \{ 1 \}$ with G_i/G_{i+1} abelian for all i;

Exercise 2.9. Let G be a finite group.
(1) Prove that the following are equivalent.
  (a) G is soluble,
  (b) there is a composition series whose factors are of prime order,
  (c) the factors of any composition series are of prime order.
(2) Prove that the following are equivalent.
  (a) G is soluble,
  (b) there is a principal series whose factors are elementary abelian.,
  (c) the factors of any principal series are elementary abelian.
(3) Prove the following.
  (a) If G is soluble and $H \leq G$, then H is soluble.
  (b) If G is soluble and $N \trianglelefteq G$, then G/N is soluble.
  (c) If G is soluble, and $\varphi : G \to K$ is a morphism, then $\varphi(K)$ is soluble.
  (d) If $N \trianglelefteq G$, and both N and G/N are soluble, then G is soluble.