DIARY OF THE COURSE
ADVANCED GROUP THEORY

A.A. 2018/19

INSTRUCTOR: ANDREA CARANTI

Lecture 1. Wednesday 27 February 2019 (2 hours)

Series and normal series in a group. Right operator groups: Ω-groups. Ω-subgroups and Ω-series. Ω-composition series: composition series and principal series.

The factors of a composition series are simple groups.

Lecture 2. Friday 1 March 2019 (2 hours)

The factors H_{i+1}/H_i of a principal series are minimal normal subgroups of G/H_i.

A minimal normal subgroup of a finite group is a characteristically simple groups.

An abelian minimal normal subgroup is an elementary abelian p-group, for some prime p.

A non-abelian minimal normal subgroup is the direct product of isomorphic copies of the same simple group.

Lemma: $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

Converse: a direct product of isomorphic copies of the same simple group is characteristically simple. (Start.)

Lecture 3. Wednesday 6 March 2019 (2 hours)

The minimal normal subgroups of a direct product of isomorphic copies of the same *nonabelian* simple group is characteristically simple are the factors.

Action of the symmetric group on S^n.

Elementary abelian groups and vector spaces.

A direct product of isomorphic copies of the same simple group is characteristically simple. (Completion.)

Uniqueness of factors of an Ω-composition series.

Commutators, derived/commutator subgroup.

Lecture 4. Friday 8 March 2019 (2 hours)

Properties of the derived subgroup. The derived subgroup is the smallest normal subgroup with an abelian quotient.

Soluble groups: equivalent statements.

Properties of soluble groups with respect to subgroups, quotient groups and extensions.

Trento, 2018/19.

In a soluble group, the quotient of a composition (resp. principal) series are cyclic of prime order (resp. elementary abelian p-groups).

Lecture 5. Wednesday 13 March 2019 (2 hours)

Nilpotent groups.

Lecture 6. Friday 15 March 2019 (2 hours)

Finite nilpotent groups. Hall-Witt identity. Three subgroups lemma.
Representations of groups: introduction.

Lecture 7. Wednesday 20 March 2019 (2 hours)

Representations of groups. From group actions to representations. Invariant spaces and subrepresentations. Direct sum of representations. Examples. Maschke's Theorem (statement).

Lecture 8. Friday 22 March 2019 (2 hours)

Proof of Maschke's Theorem. Corollary: every representation is the direct sum of irreducible ones. Morphisms of representation. The group algebra.

Lecture 9. Wednesday 27 March 2019 (2 hours)

The group algebra. Universal property of the group algebra.
Algebras.
Right and left modules.
Modules over the group algebra are equivalent to group representations.
Inner products, orthonormal bases. Orthogonal subspace.

Lecture 10. Friday 29 March 2019 (2 hours)

Maschke's Theorem revisited.
Adjoint matrices, unitary matrices.
All representations can be taken as unitary.
Every representation is a direct sum of irreducible ones.

Lecture 11. Friday 5 April 2019 (2 hours)

Orthogonality of the components of the irreducible representations.
Characters. Characters depend only on the isomorphism class of a representation. Characters are class functions.
Irreducible characters are orthonormal.
The right regular representation and its character.

Lecture 12. Wednesday 10 April 2019 (2 hours)

A character uniquely determines a representation. In particular, a character χ is the character of an irreducible representation if and only if $(\chi, \chi) = 1$. (Hence we can use the term *irreducible* character.)

Examples of the right regular representation and its character.

An irreducible character χ appears in the regular representation $\chi(1)$ times.

$$\rho = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi,$$

hence

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

Lecture 13. Wednesday 17 April 2019 (2 hours)

The isomorphism

$$\mathbf{C}[G] \cong \bigoplus_{\chi \in \mathrm{Irr}(G)} M_{\chi(1)}(\mathbf{C}).$$

Computing the centres of the two algebras, ones gets that the number of irreducible characters of a finite group G equals the number of its conjugacy classes.

As a consequence, the irreducible characters χ of a finite abelian group G are all linear, i.e. $\chi(1) = 1$.

Linear characters of finite cyclic groups.

Lecture 14. Friday 19 April 2019 (2 hours)

Another proof, via commuting matrices, that the irreducicble characters of a finite abelian group are linear.

The linear characters of finite abelian groups. The dual group.

Characters from quotients. The linear characters of a finite abelian group G are the linear characters of G/G'.

The character table of S_3

Lecture 15. Wednesday 8 May 2019 (2 hours)

The product of a linear character and a(n irreducible) character is a(n irreducible) character. (As a consequence, the zero in the non-linear character of S_3.)

Permutation representations and characters. Number of orbits as the average of fixed points. Doubly transitive groups. More on the characters of S_3.

Irreducible characters of S_4.

Lecture 16. Friday 10 May 2019 (2 hours)

Irreducible characters of S_4 and A_4. Decomposition $S_4 = VS_3$.

Integral elements and algebraic integers. A rational number which is algebraic is an integer.

The degree of an irreducible character divides the order of the group.

Lecture 17. Wednesday 15 May 2019 (2 hours)

Lemma on norms in a Galois extension with abelian Galois group.
Lemma on sums of complex numbers of norm 1, and sums of roots of unity.
A non-linear character has always a zero.

Lecture 18. Friday 17 May 2019 (2 hours)

Products of conjugacy classes. The other orthogonality relations. Size of a centraliser when passing to a quotient group.
Structure constants of the centre of the group algebra (beginning).

Lecture 19. Wednesday 22 May 2019 (2 hours)

Structure constants of the centre of the group algebra (completion).
Burnside's theorem about the solubility of groups of order $p^a q^b$.
Induced characters and Frobenius reciprocity.

Lecture 20. Friday 24 May 2019 (2 hours)

Frobenius groups: existence of the Frobenius kernel.

Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, 38123 Trento
E-mail address: andrea.caranti@unitn.it
URL: http://www.science.unitn.it/∼caranti/