TRENTO, A.A. 2021/22
GEOMETRY AND LINEAR ALGEBRA
EXERCISE SHEET # 12

Important! In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

Exercise 12.1. State and prove the Gram-Schmidt theorem.

Exercise 12.2. Let $v_1 = [1, 1]$ and $v_2 = [1, 2]$.

(1) Show that v_1, v_2 is a base for $\mathbf{R}^2$.
(2) Apply Gram-Schmidt to v_1, v_2 to find a base of $\mathbf{R}^2$ which is orthonormal with respect to the scalar product.

Exercise 12.3. Let

$$\begin{aligned} v_1 &= [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}], \\ v_2 &= [1/\sqrt{2}, 0, -1/\sqrt{2}], \text{ and} \\ v_3 &= [2, 1, 0]. \end{aligned}$$

What happens when we apply Gram-Schmidt to v_1, v_2, v_3, and why?

Exercise 12.4.

(1) Show that if V is a vector space equipped with an inner product, and $v_1, \dots, v_k$ are orthonormal, then $v_1, \dots, v_k$ are linearly independent.
(2) Let $V = \mathbf{R}^3$ be equipped with the usual scalar product as an inner product.
    (a) Show that the vectors

$$\begin{aligned} b_1 &= [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}], \\ b_2 &= [1/\sqrt{2}, 0, -1/\sqrt{2}], \text{ and} \\ b_3 &= [1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}] \end{aligned}$$

    form an orthonormal base. (HINT: First show that they are orthonormal, and then…)
    (b) Write the vector $[1, 2, 3]$ as a linear combination of b_1, b_2, b_3, using scalar products.