# TRENTO, A.A. 2021/22
# GEOMETRY AND LINEAR ALGEBRA
# EXERCISE SHEET # 10

**Important!** In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

**Note:** the exercises in red and in blue have been treated during the lectures.

**Note:** for technical reasons, sometimes in the following exercises we write something like "$e_1 + (-1)e_2$", which just means $e_1 - e_2$.

*Exercise* 10.1. Show that if $v$ is an eigenvector for the linear function $f$ with respect to the eigenvalue $\lambda$, and $a \neq 0$ is a number, then $av$ is also an eigenvector for $f$ with respect to the eigenvalue $\lambda$.

*Exercise* 10.2. Define diagonalisable matrices.

*Exercise* 10.3. Define the algebraic and geometric multiplicity of an eigenvalue.

*Exercise* 10.4.

(1) Show that eigenvectors relative to distinct eigenvalues are linearly independent.
(2) Show that if a linear function has distinct eigenvalues, then it is diagonalisable.

*Exercise* 10.5. Let $V$ be a vector space of dimension 2, and let $e_1, e_2$ be a base.

Consider the linear function $f : V \to V$ given by

$$\begin{cases} f(e_1) = 5e_1 + 2e_2 \\ f(e_2) = (-12)e_1 + (-5)e_2 \end{cases}$$

Write the matrix $A$ of $f$ with respect to the base$e_1, e_2$.

Compute the characteristic polynomial (in the variable $\lambda$)

$$\det(A - \lambda I),$$

where $I$ is the $2 \times 2$ identity matrix. Find the eigenvalues $\lambda_1, \lambda_2$ if $A$ as the solutions to the equation

$$\det(A - \lambda I) = 0.$$

For each eigenvalue $\lambda_i$, find an eigenvector $g_i$. Show that $g_1, g_2$ is a base of $V$, and write the matrix $B$ of $f$ with respect to the base $g_1, g_2$.

*Exercise* 10.6. Let $V$ be a vector space of dimension 2, and let $e_1, e_2$ be a base.

Consider the linear function $f : V \to V$ given by

$$\begin{cases} f(e_1) = (-4)e_1 + 10e_2 \\ f(e_2) = (-3)e_1 + 7e_2 \end{cases}$$

Write the matrix $A$ of $f$ with respect to the base$e_1, e_2$.

Compute the characteristic polynomial (in the variable $\lambda$)

$$\det(A - \lambda I),$$

where $I$ is the $2\times 2$ identity matrix. Find the eigenvalues $\lambda_1, \lambda_2$ if $A$ as the solutions to the equation

$$\det(A - \lambda I) = 0.$$

For each eigenvalue $\lambda_i$, find an eigenvector $g_i$. Show that $g_1, g_2$ is a base of $V$, and write the matrix $B$ of $f$ with respect to the base $g_1, g_2$.

*Exercise* 10.7. Let $V$ be a vector space of dimension $n$, and let $f : V \to V$ be a linear function. Suppose $V$ has a base $e_1, \ldots, e_n$ consisting of eigenvectors of $f$, that is, there are numbers $\lambda_1, \ldots, \lambda_n$ such that

$$f(e_i) = \lambda_i e_i$$

for $i = 1, \ldots, n$.

Show that the matrix of $f$ with respect to the base $e_1, \ldots, e_n$ is

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

*Exercise* 10.8. Compute the characteristic polynomial of $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, and show that $A$ does not have real eigenvalues.

*Exercise* 10.9. Compute the characteristic polynomial of

$$A = \begin{bmatrix} -9 & -4 \\ 25 & 11 \end{bmatrix}.$$

(1) Show that the roots of the characteristic polynomial are $1, 1$.
(2) Show that the eigenvectors with respect to the eigenvalue 1 form a subspace of dimension 1 (minus the zero vector).
(3) If $g_1$ is one of these eigenvectors, choose arbitrarily a vector $g_2$ such that $g_1, g_2$ is a base, and write the matrix of $A$ with respect to this base.

*Exercise* 10.10. For each of the following matrices $A$,

(1) compute the characteristic polynomial,
(2) compute the eigenvalues,
(3) state whether $A$ is diagonalisable or not.
    (a) If it is not diagonalisable, please stop.

(b) If it is diagonalisable, find a base of eigenvectors, and a matrix $S$ such that $S^{-1}AS$ is diagonal.

$$\begin{bmatrix} 5 & 3 \\ -4 & -2 \end{bmatrix}, \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 5/2 & -1/2 & -1 \\ 1/2 & 3/2 & -1 \\ -1/2 & -1/2 & 2 \end{bmatrix}, \begin{bmatrix} 3/2 & 0 & -1/2 \\ 1/2 & 1 & -1/2 \\ -1/2 & 0 & 3/2 \end{bmatrix},$$
$$\begin{bmatrix} 2 & -1/2 & -1/2 \\ 1 & 1/2 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 2 \\ -1 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 4 & -6 \\ -4 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}.$$

If at some point while working out the exercise you are able to say that the matrix is diagonalisable (what do I know, perhaps the eigenvalues are distinct), then spell it out!

*Exercise* 10.11. For each of the following matrices $A$,

(1) compute the characteristic polynomial,
(2) compute the eigenvalues,
(3) state whether $A$ is diagonalisable or not.
  (a) If it is not diagonalisable, please stop.
  (b) If it is diagonalisable, find a base of eigenvectors, and a matrix $S$ such that $S^{-1}AS$ is diagonal.

$$\begin{bmatrix} 3 & 4 & 2 \\ -1 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 4 & -6 \\ -4 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}, \begin{bmatrix} 8 & 8 & -2 \\ -4 & -3 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

If at some point while working out the exercise you are able to say that the matrix is diagonalisable (what do I know, perhaps the eigenvalues are distinct), then spell it out!