TRENTO, A.A. 2021/22
GEOMETRY AND LINEAR ALGEBRA
EXERCISE SHEET # 9

Important! In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

Exercise 9.1. Let $V = \mathbf{R}^2$ and $W = \mathbf{R}^2$.

(1) Prove that $\mathcal{B} = \{(1,1),(1,0)\}$ is a basis of $\mathbf{R}^2$.
(2) Consider the function $f : V \to W$ defined as
$$f(\alpha,\beta) = (\alpha - \beta, 3\beta) \quad \text{for all } (\alpha,\beta) \in V.$$
Prove that f is linear.
(3) Write the matrix M of the function f with respect to the standard basis $\mathcal{E}$ on V and W.
(4) Write the matrix N of the function f with respect to the standard basis $\mathcal{E}$ on V and $\mathcal{B}$ on W.

Exercise 9.2. Let $V = \mathbf{R}^3$ and $W = \mathbf{R}^2$.

(1) Prove that $\mathcal{B} = \{(1,1),(1,2)\}$ is a basis of W.
(2) Consider the function $f : V \to W$ defined as
$$f(\alpha,\beta,\gamma) = (\alpha - \gamma, \beta) \quad \text{for all } (\alpha,\beta,\gamma) \in V.$$
Prove that f is linear.
(3) Write the matrix M of the function with f with respect to the standard basis $\mathcal{E}$ on V and $\mathcal{B}$ on W.

Exercise 9.3. Let $V = \mathbf{R}^4$ and $W = \mathbf{R}^3$.

(1) Prove that $\mathcal{B} = \{(1,1,2),(1,0,1),(0,0,1)\}$ is a basis of $\mathbf{R}^3$.
(2) Consider the linear function $f : V \to W$ defined as
$$f(a,b,c,d) = (a + 2b, 3c, a - d) \quad \text{for all } (a,b,c,d) \in V.$$
(Optional: check that f is linear).
(3) Write the matrix M of f with respect to the standard bases on V and W.
(4) Write the matrix N of f with respect to the standard basis $\mathcal{E}$ on V and $\mathcal{B}$ on W.

Exercise 9.4. Consider the following bases of $\mathbf{R}^2$:
$$\mathcal{B} = \{(1,1),(1,2)\}, \quad \mathcal{B}' = \{(1,0),(1,1)\}.$$

(1) Find the matrix M of change of basis from $\mathcal{B}$ to the standard basis $\mathcal{E}$.
(2) Find the matrix N of change of basis from $\mathcal{E}$ to $\mathcal{B}$.
(3) Find the matrix P of change of basis from $\mathcal{B}$ to $\mathcal{B}'$.

(4) Find the coordinates of the vector $v = (3,5)$ with respect to the bases $\mathcal{B}, \mathcal{B}', \mathcal{E}$.

Exercise 9.5. Consider the following sets in $\mathbf{R}^3$:

$$\mathcal{B} = \{(1,1,0),(1,0,2),(1,0,0)\}, \quad \mathcal{B}' = \{(1,0,1),(0,1,0),(2,0,0)\}.$$

(1) Prove that $\mathcal{B}$ and $\mathcal{B}'$ are bases of $\mathbf{R}^3$.
(2) Find the matrix M of change of basis from $\mathcal{B}$ to the standard basis $\mathcal{E}$.
(3) Find the matrix N of change of basis from $\mathcal{E}$ to $\mathcal{B}$.
(4) Find the matrix P of change of basis from $\mathcal{B}$ to $\mathcal{B}'$.
(5) Find the coordinates of the vector $v = (1,4,6)$ with respect to the bases $\mathcal{E}, \mathcal{B}, \mathcal{B}'$.

Exercise 9.6. Consider the linear function $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined as $f(a,b) = (-b,-a)$ for all $(a,b) \in \mathbf{R}^2$.

(1) Write the matrix A of f with respect to the standard basis.
(2) Find the eigenvalues and a basis for each eigenspace of A.

Exercise 9.7. Consider the linear function $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined as $f(a,b) = (a+b,b)$ for all $(a,b) \in \mathbf{R}^2$.

(1) Write the matrix A of f with respect to the standard basis.
(2) Find the eigenvalues and a basis for each eigenspace of A.