TRENTO, A.A. 2021/22
GEOMETRY AND LINEAR ALGEBRA
EXERCISE SHEET # 8

Important! In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

Exercise 8.1. Consider the linear function $f : \mathbf{R}^4 \to \mathbf{R}^3$ defined as

$$f(x, y, w, z) = (x + y,\ x + y + z,\ 2x + 2y + z,\ y + 2w + 3z)$$

for all $(x, y, w, z) \in \mathbf{R}^4$.

Say if f is surjective/injective.

Exercise 8.2. Consider the linear function $f : \mathbf{R}^3 \to \mathbf{R}^4$ defined as

$$f(x, y, z) = (x - y,\ x + y + z,\ y + z, 2x + z)$$

for all $(x, y, z) \in \mathbf{R}^3$.

Say if f is surjective/injective.

Exercise 8.3. Let V be a vector space with $\dim(V) = n$. Let $v_1, \dots, v_n$ be a basis of V. Prove that the function $f : \mathbf{R}^n \to V$, defined as

$$f(c_1, \dots, c_n) = c_1 v_1 + \cdots + c_n v_n$$

for all $(c_1, \dots, c_n) \in \mathbf{R}^n$, is linear.

Exercise 8.4. Let V, W be vector spaces, and $f : V \to W$ be a linear function.

(1) Show that $\ker(f) = \{\, v \in V : f(v) = 0 \,\}$ is a subspace of V, and $F(V) = \{\, f(v) : v \in V \,\}$ is a subspace of W.

(2) Let $k_1, \dots, k_n$ be a base of $\ker(f)$, and $b_1, \dots, b_r$ be elements of V such that $b_1, \dots, b_r, k_1, \dots, k_n$ is a base of V.

(a) Show that $f(b_1), \dots, f(b_r)$ is a base of $f(V)$.

(b) Let $w_1, w_2, \dots$ be vectors such that $f(b_1), \dots, f(b_r), w_1, w_2, \dots$ is a base of W. Show that with respect to these bases of V and W the matrix of f is of the block form

$$\left[\begin{array}{c|c} I & 0_{r,n} \\ \hline 0 & 0 \end{array}\right]$$

Here I is an $r \times r$ identity matrix, $0_{r,n}$ is a zero $r \times n$ matrix, and the other zeroes are appropriate zero matrices.

Exercise 8.5. Define eigenvalues and eigenvectors.