TRENTO, A.A. 2021/22
GEOMETRY AND LINEAR ALGEBRA
EXERCISE SHEET # 5

Important! In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

Exercise 5.1.

(1) Define
   (a) systems of generators,
   (b) linear dependence,
   (c) linear independence,
   (d) bases.

(2) Show that the following are equivalent.
   (a) $v_1, \dots, v_m$ are linearly dependent, and
   (b) (at least) one of the v_i can be written as a linear combination of the others.

   Show that as a consequence, if $v_1, \dots, v_m$ are a system of generators of a space V, and they are linearly dependent, say v_m is a linear combination of $v_1, \dots, v_{m-1}$, then $v_1, \dots, v_{m-1}$ is a system of generators for V, that is, you can safely drop v_m.

(3) Show that the following are equivalent (to being a base).
   (a) $v_1, \dots, v_m$ are a system of generators of a space V, which are linearly independent, and
   (b) Every element of V can be written *uniquely* as a linear combination of $v_1, \dots, v_m$.

Exercise 5.2. Show that if one of $v_1, \dots, v_m$ is zero, then $v_1, \dots, v_m$ are linearly dependent.

Exercise 5.3. For each of the following sysyem of vectors, state whether they are a basis of $\mathbf{R}^3$. When they are not, state explicitly which condition fails.

(1)

$$\begin{cases} v_1 = (1, 0, 0) \\ v_2 = (0, 1, 0) \\ v_3 = (0, 0, 1) \end{cases}$$

(2)

$$\begin{cases} v_1 = (1, 0, 0) \\ v_2 = (0, 1, 0) \end{cases}$$

(3)

$$\begin{cases} v_1 = (1, -1, 0) \\ v_2 = (0, 1, -1) \\ v_3 = (-1, 0, 1) \end{cases}$$

(4)

$$\begin{cases} v_1 = (1, -1, 0) \\ v_2 = (0, 1, -1) \\ v_3 = (1, 0, 1) \end{cases}$$