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Note. The description of the content of a yet to be delivered lecture is meant to be part of the planning.

Lecture 1. Monday 13 September 2021 (1 hr)

One linear equation in one unknown. Discussion of the various cases.

One linear equation in two unknowns. Beginning of the discussion of the various cases.

Lecture 2. Wednesday 15 September 2021 (2 hrs)

Manipulating logical operators.

One linear equation in two unknowns: discussion.

Two linear equations in two unknowns. The substitution method.

Applied vectors and vectors in the plane $\mathbf{R}^2$. Length of a segment and a vector.

Multiplication of a vector by a scalar: geometrical meaning.

Parametric equation of a line in the plane through the origin.

Lecture 3. Monday 20 September 2021 (2 hrs)

Sum of vectors: the parallelogram rule.

Parametric equation of a line in the plane through a point.

Cartesian/implicit equation of a line in the plane.

Geometrical interpretation of the solutions of a linear system of 2 linear equations in 2 unknowns (no solution, unique solution, infinitely many solutions) as the relative position of two lines in the plane.

Scalar product and the angle between two vectors.

Lecture 4. Wednesday 22 September 2021 (2 hrs)

If $(a, b) \neq (0, 0)$, then the line given by the equation

$$ax + by = c$$

is orthogonal to the vector (a, b).

Line through two points.

Given a line $\mathfrak{l}$, and a point p, how to find

(1) the line $\mathfrak{r}$ through the point p, orthogonal to $\mathfrak{l}$, and
(2) the orthogonal projection of p onto $\mathfrak{l}$.

Cartesian equation of a plane in $\mathbf{R}^3$. Parametric equations (introduction).

Date: Trento, A. A. 2021/22.

Lecture 5. Monday 27 September 2021 (2 hrs)

Parametric equations of a plane.

Vector spaces, subspaces.

Systems of (homogeneous) linear equations. Subspaces are exactly the sets of solutions of systems of homogeneous linear equations (no proof for the time being).

Gauss' reduction (beginning).

Lecture 6. Wednesday 29 September 2021 (2 hrs)

Gauss' reduction algorithm. Elementary operations.

Lecture 7. Monday 4 October 2021 (2 hrs)

Vector spaces: the $\mathbf{R}^n$ and their subspaces, that is, the non-empty subsets that are closed for linear combinations.

Subspaces are exactly the sets of solutions of systems of homogeneous linear equations.

Bases of a vector space.

If a system $AX = b$ of linear equations has a solutions, then its solutions of are of the form $x_0 + u$, where x_0 is a particular solution, and u is a solution of the associated homogeneous system $AX = 0$.

Lecture 8. Wednesday 6 October 2021 (2 hrs)

Finding a particular solution of a (non-homogeneous) system. Rouché-Capelli: ranks of the incomplete and complete matrices.

Finding a basis for the space of solutions of a homogeneous system.

Lecture 9. Monday 11 October 2021 (2 hrs)

The span of a system of vectors is a subspace. Systems of generators of a vector space. Linearly independent vectors. Checking linear independence is equivalent to checking that a system of homogeneous linear equations has only the trivial solution. A base is a system of generators which are linearly independent.

All bases of a vector space have the same number of elements. Dimension of a vector space. From a system of generators, one can extract a base. Given a system of linearly independent vectors, one can add vectors to obtain a base.

Lecture 10. Wednesday 13 October 2021 (2 hrs)

Rouché-Capelli: the complete guide to find the solutions to a system of linear equations, by determining whether there is a solution, finding a particular one in case, and determining a basis of the space of solutions of the associated system of homogeneous linear equations.

Invertible matrices. Determinant of a matrix.

Lecture 11. Monday 18 October 2021 (2 hrs)

Properties of the determinant, and how to compute it. The case of 2×2 matrices. Laplace expansion. The case of 3×3 matrices. Invertible matrices.

Lecture 12. Wednesday 20 October 2021 (2 hrs)

(Lecture held by Simone Ugolini.)

Extension of a set of linearly independent vectors to a basis. Extraction of a basis from a set of generators. If $S = \{v_1, \dots, v_n\}$ is a set of n linearly independent vectors of a vector space V with $\dim(V) = n$, then S is a basis of V. If W is a subspace of a vector space V with $\dim(V) = n$, then $\dim(W) \leq n$. If $\dim(W) = 0$, then $W = \{O\}$. If $\dim(W) = n$, then $W = V$.

Lecture 13. Monday 25 October 2021 (2 hrs)

Linear functions. Writing linear functions with respect to a basis. Composition of functions corresponds to multiplication of matrices.

Lecture 14. Wednesday 27 October 2021 (2 hrs)

(Lecture held by Simone Ugolini.)

Column space and null space of a matrix. Rank-nullity theorem. Extension of a set of linearly independent vectors to a basis using an auxiliary matrix and finding a basis of its column space. Exercises on matrices associated with linear functions with respect to assigned bases.

Lecture 15. Wednesday 3 November 2021 (3 hrs)

Intermediate test (midterm).

Lecture 16. Monday 8 November 2021 (2 hrs)

(Lecture held by Simone Ugolini.)

Kernel ($\ker(f)$) and image ($\mathrm{im}(f)$) of a linear function $f : V \to W$. If $f : V \to W$ is linear, then $\ker(f)$ is a vector subspace of V and $\mathrm{im}(f)$ is a vector subspace of W. A linear function f is injective if and only if $\ker(f) = \{O_V\}$. Isomorphic vector spaces. Rank-nullity theorem for linear functions. If $f : V \to W$ is a linear function and $n = \dim(V)$, then the following hold.

(1) If f is an isomorphism, then $\dim(W) = n$.
(2) If $\dim(W) = n$, then the following are equivalent:
    (a) f is an isomorphism.
    (b) f is injective.
    (c) f is surjective.

Exercise.

Lecture 17. Wednesday 10 November 2021 (2 hrs)

The kernel $\ker(f) = \{ v \in V : f(v) = 0 \}$ of a linear function $f : V \to W$ is a subspace of V.

Let V, W be vector spaces, and $f : V \to W$ a linear function. Then there are bases of V and W such that the matrix of f with respect to these bases has the block form

$$\left[\begin{array}{c|c} I & 0_{r,n} \\ \hline 0 & 0 \end{array}\right]$$

Here I is an $r \times r$ identity matrix, where r is the rank of f, that is, the dimension of the image subspace $f(V)$, and $0_{r,n}$ is a zero $r \times n$ matrix, where $n = \dim(\ker(f))$ is the nullity of f.

Let V be a vector space, and $f : V \to V$ a linear function. Suppose there is a base $v_1, \dots, v_n$ of V such that the matrix of f with respect to this base is of the form

$$\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

for some $a_1, \dots, a_n \in \mathbf{R}$. Then we have $f(v_i) = a_i v_i$ for all i. It follows that V has a base made of eigenvectors, according to the following definition.

Let V be a vector space, and $f : V \to V$ a linear function. A number a is said to be an *eigenvalue* of f if there is $0 \neq v \in V$ such that $f(v) = av$. Any such v is said to be an *eigenvector* for f with respect to the eigenvalue a.

An eigenvector with respect to the eigenvalue 0 is just a non-zero element of the kernel of f.

A number a is an eigenvalue of f if and only if $\det(B - aI) = 0$, where B is the $n \times n$ matrix of f with respect to an arbitrary base (so $n = \dim(V)$ and I is the $n \times n$ identity matrix).

Lecture 18. Monday 15 November 2021 (2 hrs)

(Lecture held by Simone Ugolini.)

If a is an eigenvalue for a linear function $f : V \to V$, then the set $E_a = \{v \in V : f(v) = av\}$ is said to be the *eigenspace* of the eigenvalue a. The eigenspace E_a is a vector subspace of V. If A is the $n \times n$ matrix of the function f with respect to a basis $\mathcal{B}$, then the eigenvalues of f are the roots of the *characteristic polynomial* $p(\lambda) = \det(A - \lambda I)$. The characteristic polynomial of f has degree n. Eigenvalues, eigenspaces and eigenvectors of a $n \times n$ matrix are referred to the linear function $f : \mathbf{R}^n \to \mathbf{R}^n$ defined as $f(x) = Ax$ for any column vector $x \in \mathbf{R}^n$. Eigenvalues and eigenspaces of the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Geometric interpretation of the linear functions induced by the matrix A_2 (shear mapping), A_3 (reflection about the line $y = x$) and A_4 (90° counterclockwise rotation). The rotation in the plane with center O and through an angle ϑ is described by the function $f(x,y) = R_\vartheta \begin{bmatrix} x \\ y \end{bmatrix}$, where

$$R_\vartheta = \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{bmatrix}.$$

If $\vartheta \neq k\pi$, for some $k \in \mathbf{Z}$, then R_ϑ has no real eigenvalue.

If V is a vector space with $\dim(V) = n$ and $\mathcal{B}, \mathcal{B}'$ are two bases of V, then the matrix M of the identity function with respect to the basis $\mathcal{B} = \{v_1, \dots, v_n\}$ on the domain and $\mathcal{B}' = \{v'_1, \dots, v'_n\}$ on the codomain is the matrix of change of basis from $\mathcal{B}$ to $\mathcal{B}'$. The inverse matrix M^{-1} is the matrix of change of bases from the basis $\mathcal{B}'$ to the basis $\mathcal{B}$. If v is a vector of V, then

$$\begin{aligned} v &= x_1 v_1 + \cdots + x_n v_n \quad \text{for some } x_1, \dots, x_n \in \mathbf{R}, \\ v &= x'_1 v'_1 + \cdots + x'_n v'_n \quad \text{for some } x'_1, \dots, x'_n \in \mathbf{R}, \end{aligned}$$

and

$$\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Lecture 19. Wednesday 17 November 2021 (2 hrs)

Matrices of a change of bases, again. An example.

Matrix of a linear function with respect to different bases (using also the matrix of change of bases).

The characteristic polynomial of a linear function does not depend on the basis.

A 2×2 example with the matrix

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Lecture 20. Monday 22 November 2021 (2 hrs)

(Lecture held by Simone Ugolini.)

Eigenvalues and eigenspaces of the matrices

$$A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

Review on roots of polynomials, Ruffini's rule and integer roots of polynomials with integer coefficients. Endomorphisms. An endomorphism on a vector space V is said to be diagonalizable if there exists a basis of V formed by eigenvectors of f. A real square matrix A is diagonalizable if there exists an invertible real matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. An endomorphism f on V is diagonalizable if and only if the matrix of f with respect to any basis of V is diagonalizable.

Lecture 21. Wednesday 24 November 2021 (2 hrs)

Eigenvectors relative to distinct eigenvalues are independent. So if the roots of the characteristic polynomial are distinct, the linear function can be diagonalised.

But: if the eigenvalues are *not* distinct, the linear function could be diagonalisable or not.

Algebraic and geometric multiplicity of an eigenvalue. If k is the algebraic multiplicity of an eigenvalue α and d is its geometric multiplicity, then $1 \le d \le k$.

A linear function is diagonalisable if and only if the two multiplicities coincide for every eigenvalue.

Examples.

Lecture 22. Monday 29 November 2021 (2 hrs)

Exercises on eigenvectors, eigenvalues, algebraic and geometric multiplicties.

Lecture 23. Wednesday 1 December 2021 (2 hrs)

The spectral theorem: a real, symmetric matrix has real eigenvalues.

Lecture 24. Monday 6 December 2021 (2 hrs)

(Lecture held by Simone Ugolini.)

Jordan canonical form of a square matrix. Real symmetric matrices are diagonalizable. Diagonalization of the matrix

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Lecture 25. Monday 13 December 2021 (2 hrs)

Reminder on the scalar product on $\mathbf{R}^n$, and inner products on a finite-dimendional vector space.

Orthonormal bases. If we have an inner product on the vector space V, and an orthonormal base $\mathcal{B}$ of V, then

(1) one can write at once a vector v as a linear combination of the elements $b_1, \dots, b_n$ of $\mathcal{B}$ as

$$v = \sum_{i=1}^{n} \langle\, v, b_i \,\rangle\, b_i,$$

and

(2) the inner product is just the scalar product of the coordinates of vectors with respect to the base $\mathcal{B}$: if $u = \sum_{i=1}^{n} x_i b_i$, $v = \sum_{i=1}^{n} y_i b_i$, then

$$\langle\, u, v \,\rangle = \sum_{i=1}^{n} x_i y_i.$$

Gram-Schmidt theorem: the algorithm.

Examples.

Lecture 26. Wednesday 15 December 2021 (2 hrs)

(Lecture held by Simone Ugolini.)

If $S = \{v_1, \dots, v_k\}$ if a set of non-zero mutually orthogonal vectors in $\mathbf{R}^n$, then the vectors in S are linearly independent. Review exercises on orthonormal bases, matrices of change of basis, diagonalizable matrices.

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